Subverting Mathematical Expectations

Christopher Rath

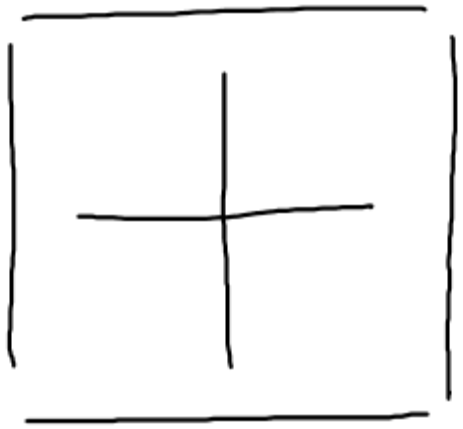
Cheadle Hulme High School

Topics that could appear today…

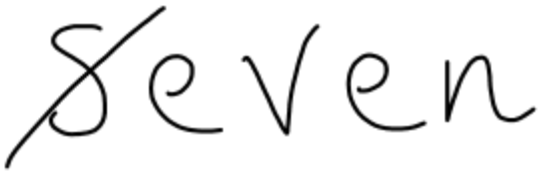
- Partial Fractions
- Number Theory
- Normal Distribution
- Matrices
- Graph Sketching
- Modulus Function
- Binomial Series
- Trigonometry
- Exponentials & Logarithms
- Vectors
- Maclaurin Series
- Parametric Equations
- The Unit Circle
- Constructing Triangles
- Surds
- Factorisation
- Shape Transformations
- Fraction Calculations
- Estimation
- Functions
- Recurrence Relations
- Proof By Contradiction
- Pythagorean Theorem
- Circle Theorems

Subverting Mathematical Expectations

“What does ‘1+1=’?”



“How do you make 7 even?”



Express in terms of partial fractions

$$\frac{4x + 12}{3x^2 + 2x - 21}$$

$X \sim N(3, 1.5^2)$

$P(2.6 < X < k) = 0.36$

$k = ?$

$$M = \begin{pmatrix} 7 & 21 \\ -2 & -6 \end{pmatrix}$$

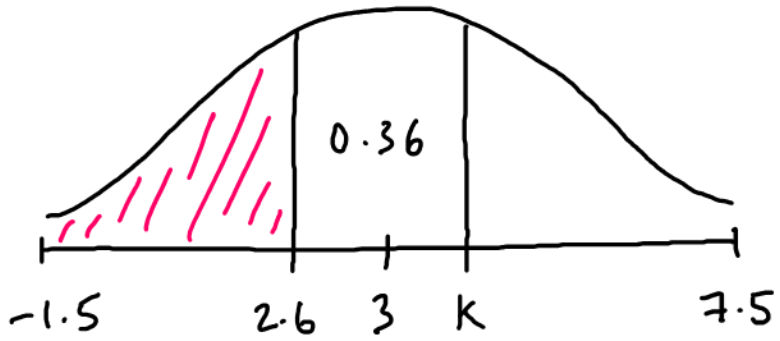
$M^2 = ?$

Solve the linear congruence

$$6x \equiv 4 \; mod \; 15$$

Express in terms of partial fractions

$$\frac{4}{3x-7}+\frac{0}{x+3}$$



k = 4.03480893

$$M^2 = \begin{pmatrix} 7 & 21 \\ -2 & -6 \end{pmatrix}$$

Solve the linear congruence

$$6x \equiv 4 \; mod \; 15$$

no solutions

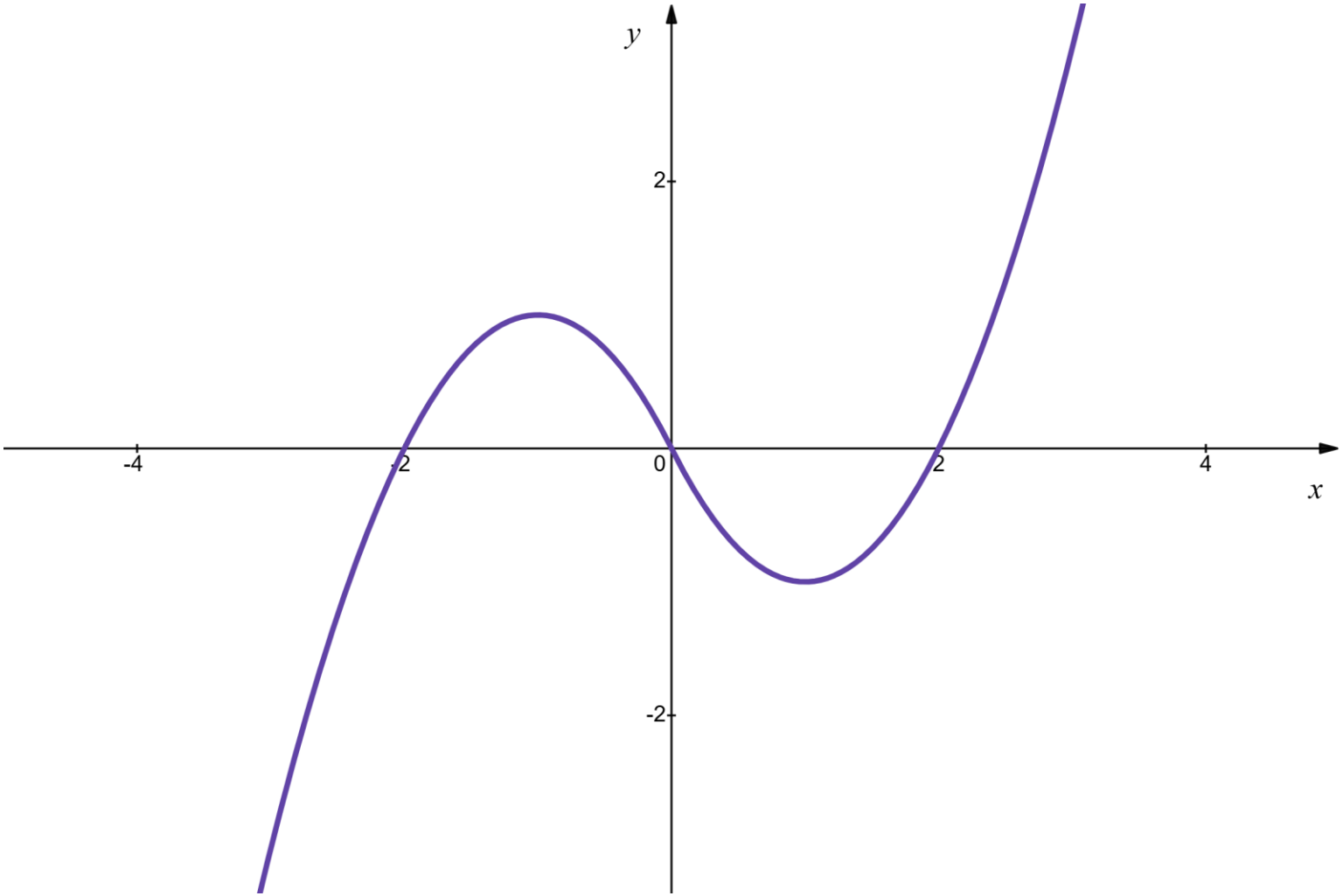
“Please expect me to be deliberately mathematically ambiguous…”

Subverting Mathematical Expectations

- Introducing a topic
- To connect ideas
- Using known results
- At KS3/4
- Enabling student ingenuity
- Appearing in exam questions
- For student intrigue
- Developing teachers’ self-efficacy

What is the equation of the curve?

https://www.desmos.com/calculator/zn4ih0gbf2



christopher.rath@chhs.org.uk

Introducing a topic

- Binomial Series
- Partial Fractions
- Inverse Of A 2x2 Matrix

Introducing a topic

- Binomial Series

"Write down the converging geometric series with first term 1 and common ratio $-x$…"

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

"Differentiate that infinite series to the 'next one'…"

$$-(1+x)^{-2} = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots$$

$x \to -1$ $\qquad$ $x \to -1$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$

"Can we predict what the next series by pattern spotting with the coefficients?"

triangle numbers!

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots$$

"How can we connect what we have done to the given formula?"

"Given what we know for $(1+x)^{-1}$, how can we work backwards to get $(1+x)^{-\frac{1}{2}}$?"

$$(1+x)^{-\frac{1}{2}} = 1 - x + x^2 - x^3 + \cdots$$

$(1+x)^{-\frac{1}{2}}$

$\times$	1	$-\frac{1}{2}x$	$+\frac{3}{8}x^2$	$-\frac{5}{16}x^3$	$+\cdots$
1	1	$-\frac{1}{2}x$	$\frac{3}{8}x^2$	$-\frac{5}{16}x^3$	
$-\frac{1}{2}x$	$-\frac{1}{2}x$	$\frac{1}{4}x^2$	$-\frac{3}{16}x^3$		
$\frac{3}{8}x^2$	$\frac{3}{8}x^2$	$-\frac{3}{16}x^3$			
$-\frac{5}{16}x^3$	$-\frac{5}{16}x^3$				

Introducing a topic

- Partial Fractions

When students can successfully deal with, e.g., $\frac{19x + 55}{(x-3)(x+4)(x+1)}$

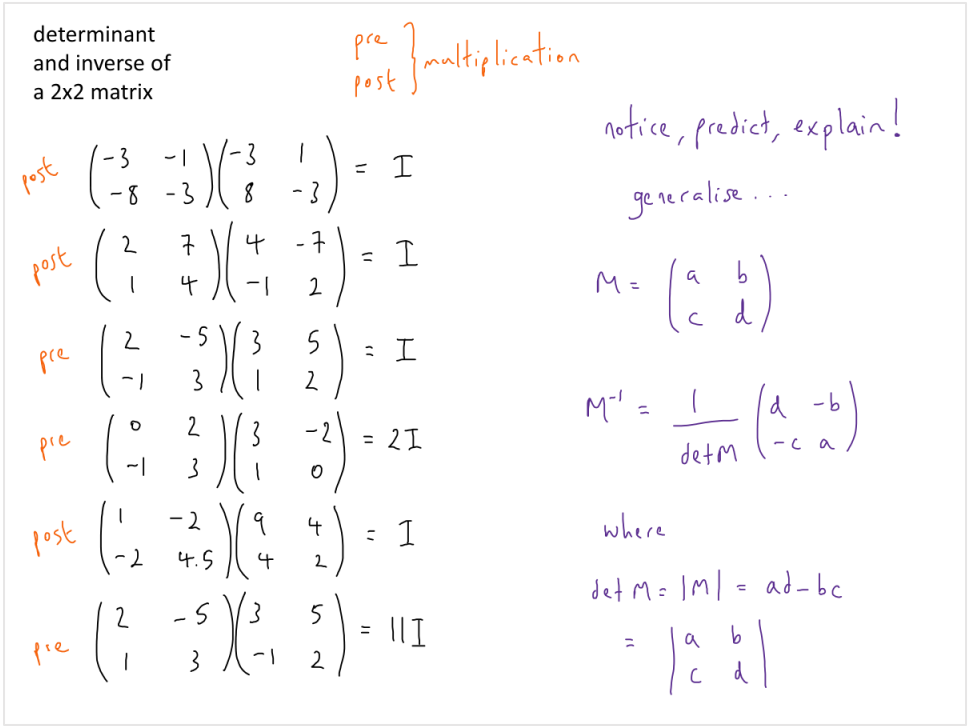
How much guidance do you give the first time they work with a repeated factor?

What happens if you do in fact try it like this...

$$\frac{3x+5}{(x+1)^2(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+1} + \frac{C}{x+2}$$

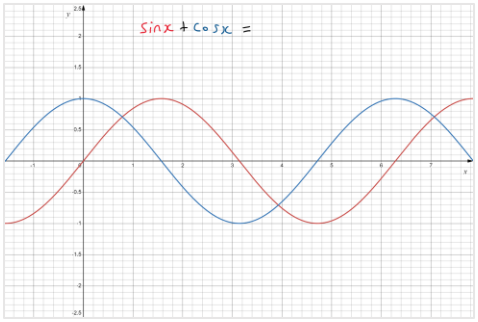
Introducing a topic

- Inverse Of A 2x2 Matrix



To connect ideas

- Trigonometric Graphs



- Exponentials & Logarithms

What if…

- The number halfway between 1 and 25 is 5
- The number halfway between 10 and 1000 is 100

What number is halfway between ____ and ____?

- Vector Product

① Work out the determinant of

$$M = \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ -2 & 2 & 3 \\ 1 & -4 & 1 \end{pmatrix}$$

② With [redacted] your calculator, evaluate

$$\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$$

③ Verify that $-2\underline{i} + 2\underline{j} + 3\underline{k}$ and $\underline{i} - 4\underline{j} + \underline{k}$ are both perpendicular to

$$28\underline{i} + 10\underline{j} + 12\underline{k}$$

Using known results

- Maclaurin Series

Use the standard result ...

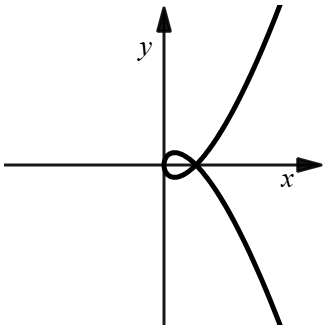
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots (-1 < x \leq 1)$$

... to approximate $\ln(2+x)$

- Sketching Parametric Graphs

Given...

$x = t^2 \qquad y = t^3 - t$



Sketch/Describe...

$x = t^2 + 1$, $y = t^3 - t$

$x = t^2$, $y = t^3 - 2t$

$x = t^2$, $y = t^3 + t$

$x = (t+1)^2$, $y = t^3 - t$

$x = t^3$, $y = t^3 - t$

Using known results

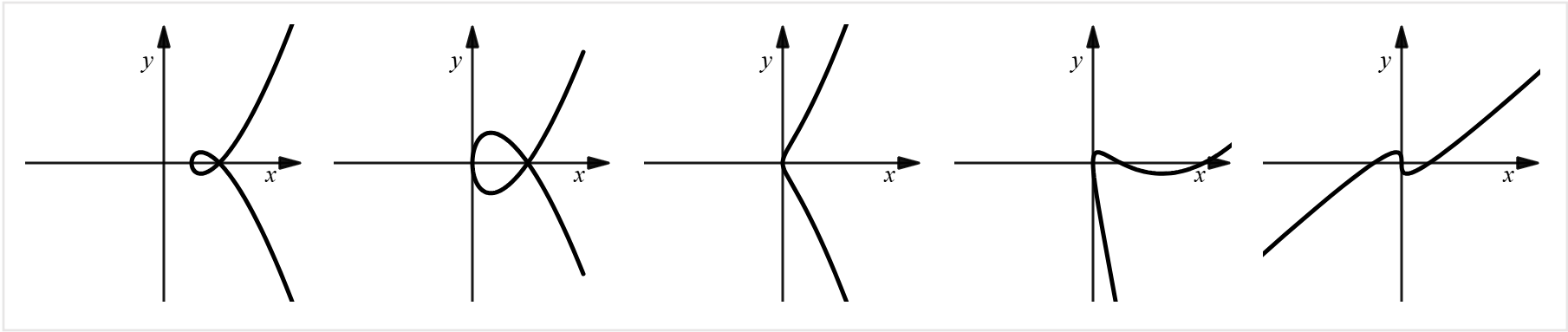
- Maclaurin Series

$\ln(2+x)$

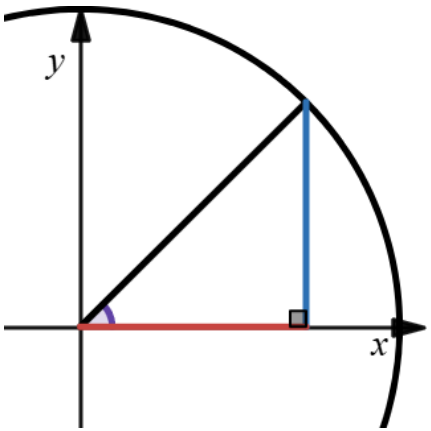
$\ln\left(2\left(1+\frac{x}{2}\right)\right) = \ln 2 + \ln\left(1+\frac{x}{2}\right)$

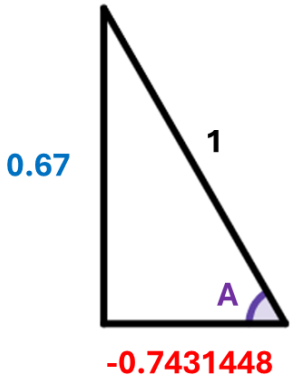
$\ln\left(1+(1+x)\right) = \ldots$

- Sketching Parametric Graphs



At KS3/4





- The Unit Circle

- Constructing Triangles

"Construct a triangle with the measurements…"

A	72°	90°	35°	100°
S	3cm	4cm	5cm	6cm
A	75°	90°	140°	121°

- Surds

Simplify…

$\sqrt{156}$ $\sqrt{155}$ $\sqrt{154}$ $\sqrt{151}$

Enabling student ingenuity

- Factorisation

(b) Factorise fully.

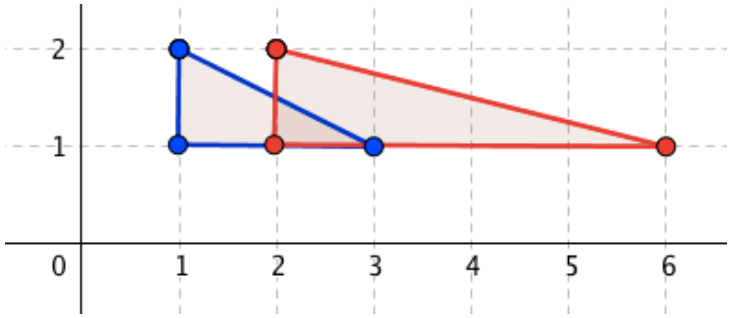
$4x^2 - 12x$

..... **[2]**

- Transformations

Describe the transformation(s)

from the object to the image



- Fraction Calculations

"Place a number in the box to make one easy, one medium and one difficult question…"

$\frac{4}{15} + \frac{1}{\square}$

Appearing in exam questions

- Binomial Series

(b) **(i)** Expand $(1+3x)^{0.5}$ as far as the term in x^3. **[3]**

(ii) State the range of values of x for which your expansion is valid. **[1]**

A student suggests the following check to determine whether the expansion obtained in part **(b)(i)** may be correct.

"Use the expansion to find an estimate for $\sqrt{103}$, correct to five decimal places, and compare this with the value of $\sqrt{103}$ given by your calculator."

(iii) Showing your working, carry out this check on your expansion from part **(b)(i)**. **[3]**

- Functions

"Sir, there was this great question about inverse functions.
I know I got it but I reckon loads of people won't have."

Appearing in exam questions

- Binomial Series

(b)	**(i)**	$1 + 0.5\times 3x + \frac{0.5\times(-0.5)}{2}\times(3x)^2$ $+ \frac{0.5\times(-0.5)(-1.5)}{3!}\times(3x)^3$ $= 1 + \frac{3}{2}x - \frac{9}{8}x^2 + \frac{27}{16}x^3$ or $1 + 1.5x - 1.125x^2 + 1.6875x^3$
(b)	**(ii)**	$-\frac{1}{3} < x < \frac{1}{3}$
(b)	**(iii)**	Sub $x = 0.01$ in their expansion gives $\sqrt{1.03} = 1.014889\ldots$ From series $\sqrt{103} = 10.14889(188\ldots)$ From calculator $\sqrt{103} = 10.14889(157\ldots)$ (Hence expansion may be correct)

- Functions

something like ...

given $f(x) = (x-2)^2 + 3$, $x \leq 2$

what is $f^{-1}(x)$?

For student intrigue

- Recurrence Relations

$$u_{n+1} = \frac{4(u_n - 2)}{u_n}, \quad u_1 = 5$$

- Trigonometric Identities

$$\cos^2\theta + \sin^2\theta \equiv 1$$

$$\cos^2\theta - \sin^2\theta \equiv \cos 2\theta$$

$$\cosh^2\theta - \sinh^2\theta \equiv 1$$

Developing teachers' self-efficacy

- Proof By Contradiction

"…is an A Level topic"

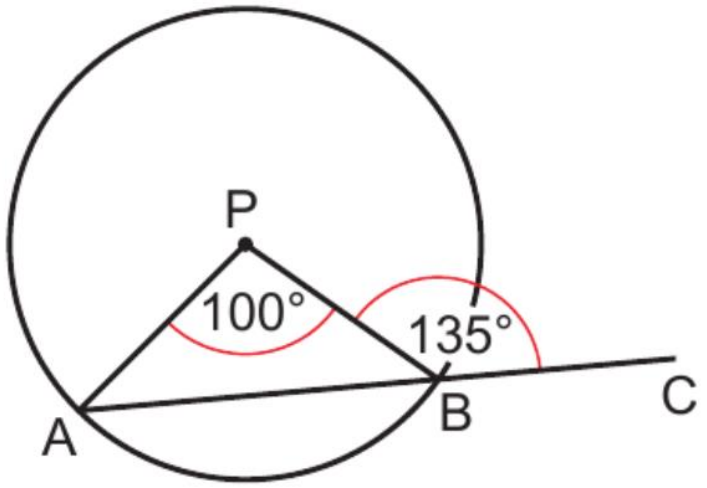
Claim:
Triangle PQR is not right-angled.
Is this correct?

$PQ = 2^2 \times 29$

$QR = 2^2 \times 3^2 \times 7$

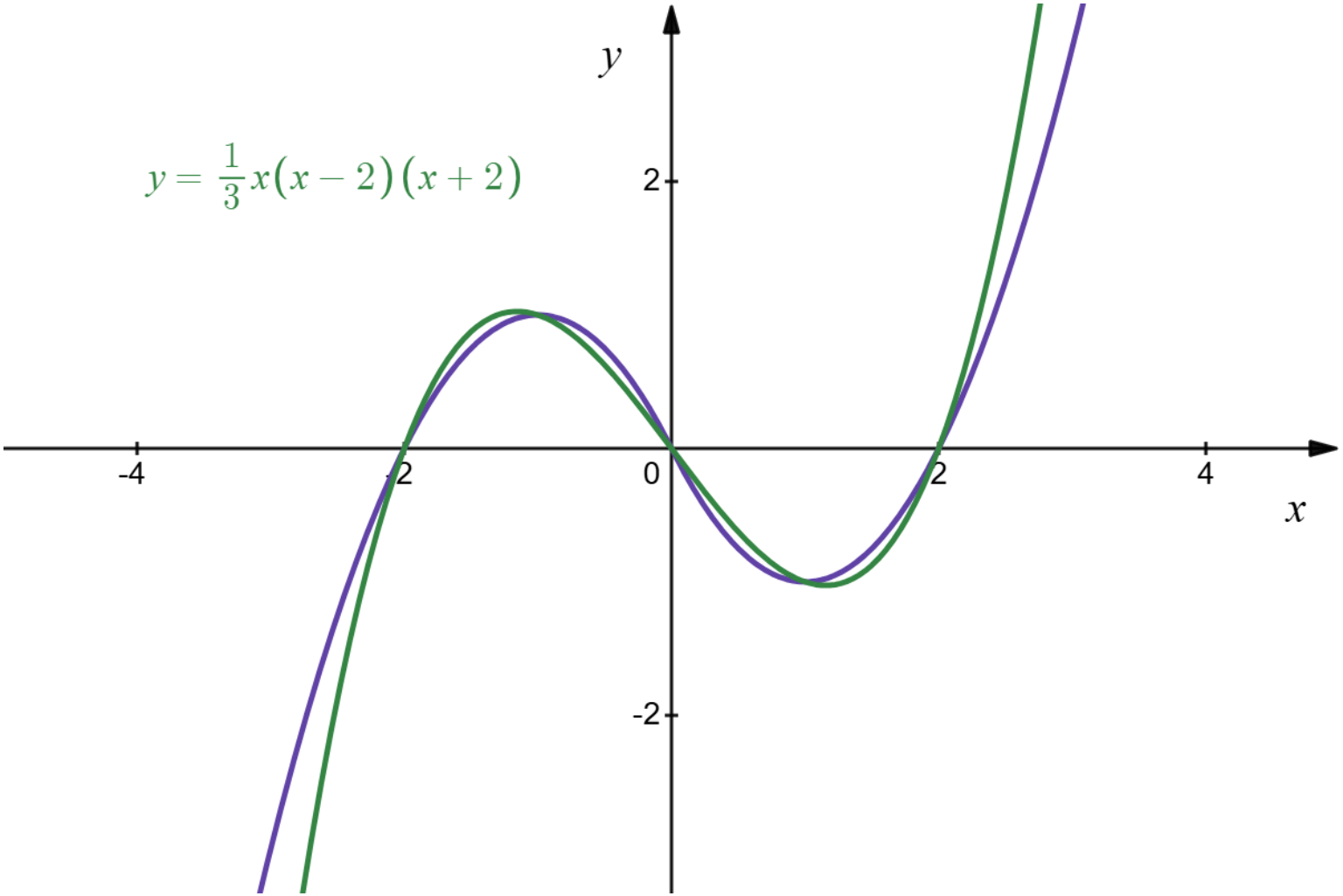
$RP = 2^2 \times 3 \times 23$

Prove that P is not the centre of the circle



What is the equation of the curve?

https://www.desmos.com/calculator/zn4ih0gbf2



Subverting Mathematical Expectations
christopher.rath@chhs.org.uk

Subverting Mathematical Expectations

Christopher Rath

christopher.rath@chhs.org.uk