Resources at the end of the PDF

MEI®

Because maths matters!

Six favourite problems

Phil Chaffé and
John Brennan-Rhodes

Couldn't share starter problem for copyright reasons

Search "No Yoshigahara favourite problem"

Why did we make this session?

- Because problems present more learning opportunities than just finding the answer.
- Develops oracy and mathematical vocabulary.
- Because it promotes deeper mathematical thinking.

…and because we like solving problems.

Six favourite problems

Because maths matters!

Have a go

- Have a go at the six problems on your tables
- Just choose the one that you fancy first, there is no order.
- If you have seen one before, try to not spoil the surprise!

While you do:

- Chat with other delegates about your different approaches
- What wider problem-solving strategies have you found?

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Tetromino problem

The problem

There are 7 possible tetrominos if you allow rotation but not reflection.

Since they combine to give $4 \times 7 = 28$ squares, can you fit one of each to cover a 4×7 grid without overlap or gaps?

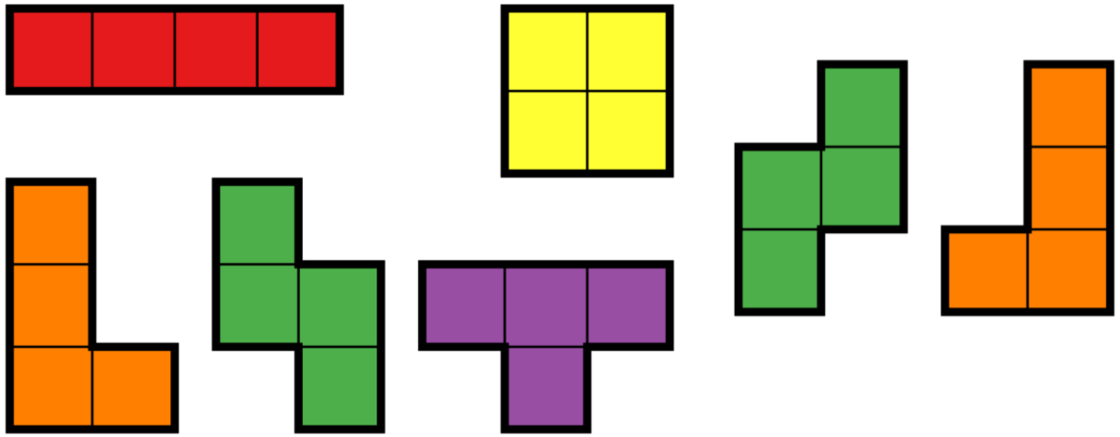


Image under creative commons
Tetromino Tiling 5x8.svg *by* R. A. Nonenmacher.

Why I chose it?

This is an example of a problem which revolves around the question:

"Do I think it is true?"

Inevitably it is worth trying based on the assumption that it *is* possible, but when should you switch to thinking it might not be?

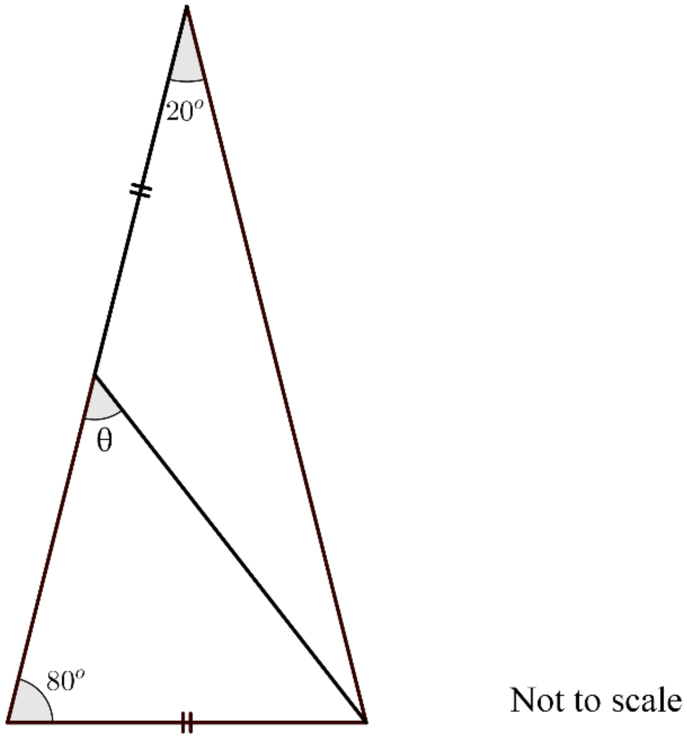
Then, how can you prove it isn't?

Finally, the solution is so beautiful, and applicable to other problems that involve a parity approach, that it really is one of my favourite problems.

80-20-80 problem

The problem

Find the value of θ.



Why I chose it?

JBR: This problem is lovely because it is so approachable. GCSE students can certainly have a go.

It is also one that can take a very, very long time as there are so many possible approaches.

In my experience, it is also easy to convince yourself you have solved it when you have actually made a small logical error.

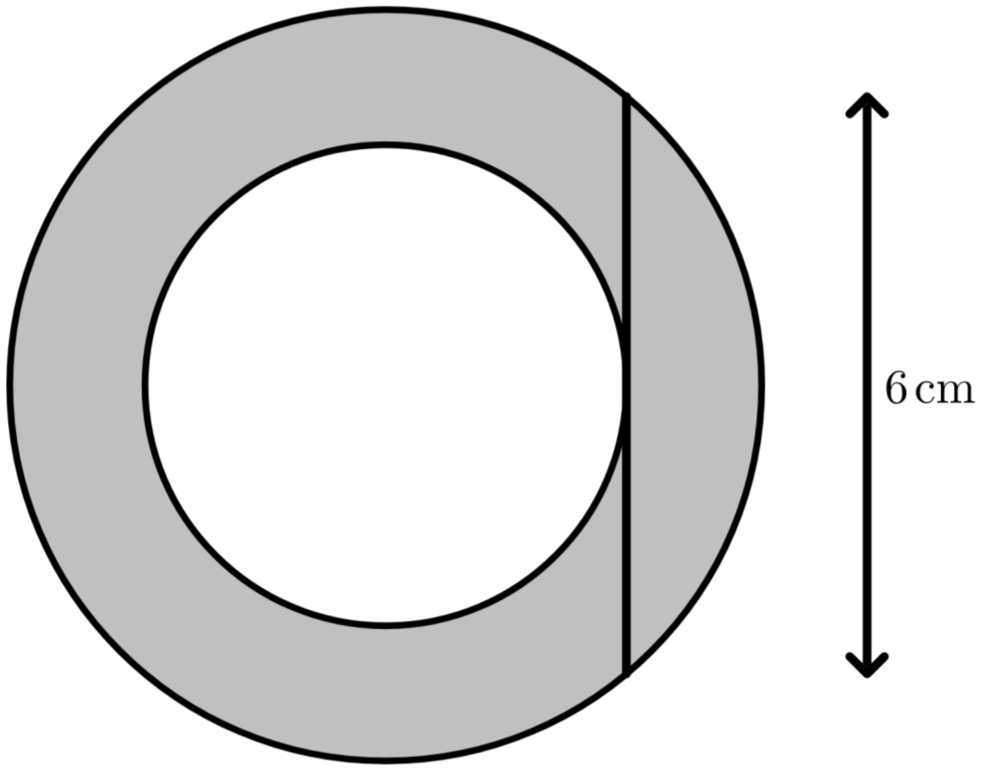
Finally, there are several very different valid solutions (although all revolve around equilateral triangles and the fact that $80 - 20 = 60$).

Always require a full proof from students!

Annulus problem

The problem

Find the shaded region



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JBR: This problem isn't that tricky, but highlights an interesting strategy in problem solving.

An observation you might make is that you aren't given the radii of the two circles.

What does this mean?

Why do we not need them?

Are they fixed but we don't know them?

In the end, you end up discovering an interesting property of an annulus, and you might even spot a remarkable shortcut to the solution!

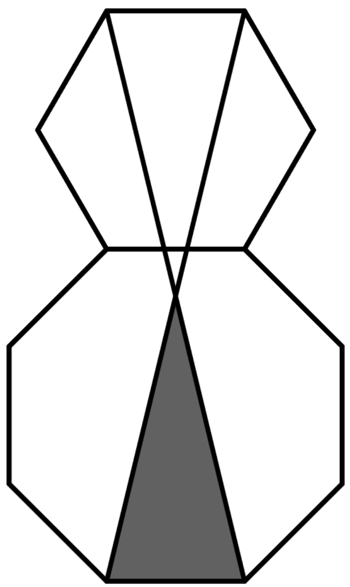
Octagon-hexagon problem

The problem

The regular hexagon and the regular octagon shown share a common edge.

The length of that edge is 2 cm.

Find the shaded area.



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PC: I wrote this problem several years ago after spotting a nice property about the sum of the height of a hexagon and the height of an octagon. I wanted to disguise this result a little, hence the shaded triangle.

The thing I like about this is the very nice expression you get for the answer.

This could easily be set using a general side-length.

It is a problem that can illustrate how useful it is to know the sine and cosine of some standard angles.

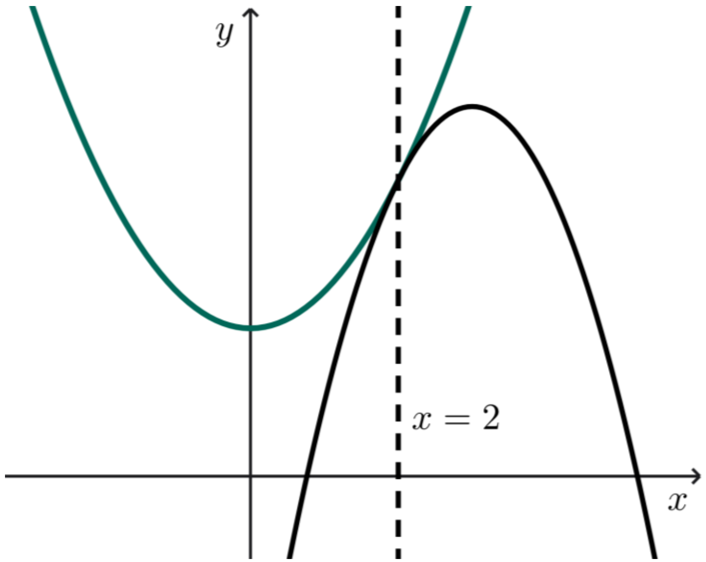
Common tangent problem

The problem

Two parabolas share a common tangent at $x = 2$.

The vertex of one of the parabolas is at (0,2) and the vertex of the other is at (3,5).

Find the equation of each parabola and the equation of the common tangent.



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Incredible indices problem

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Find the solutions for x to the equation below:

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This problem makes a clear point that you should always make sure you have found all of the solutions.

Reflection

The four stages of problem solving

Problem solving support from AMSP

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Sept	Dec	Mar	Jun	Sept	Dec	Mar	Jun
Year 12				Year 13			

In person

02 Problem Solving Matters

Jun Y12 – Sept Y13

Live Online

01 Setting Out in Problem Solving

Throughout Y12

04 TMUA Live Online

Sept - Oct Y13 and Oct -Nov Y13

05 STEP Online

Feb Y13 – Apr Y13

On Demand

03 TMUA On Demand

Self directed at your pace!

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- 7 live online weekly sessions with tutors
- Accompanying resources and support
- TMUA courses in September and another in November each year

STEP online course – Year 13

- 10 online sessions
- Accompanying resources and support
- From Feb

Applications open around November each year

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Register (for free) and find the magazine online each month.

https://amsp.org.uk/sums-steps-to-university-for-mathematical-students/

QR code for SUMS magazine

MEI run courses.

MEI

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- Six partnership universities below.
- Three in person study days
- 580 current Year 12 student places
- A mentor assigned to you who currently studies maths at university
- Follow up lectures on TMUA

The study days:

- Two in June/July, one in September
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- Workgroups with university student mentors

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Not all nice problems are puzzles

x, y and z are real numbers.

$\lfloor x \rfloor$ denotes the largest integer that satisfies $\lfloor x \rfloor \leq x$. $\{x\}$ denotes the fractional part of x so that $x = \lfloor x \rfloor + \{x\}$ and $0 \leq \{x\} < 1$.

For example, if $x = 4.2$, then $\lfloor x \rfloor = 4$ and $\{x\} = 0.2$ and if $x = -4.2$, then $\lfloor x \rfloor = -5$ and $\{x\} = 0.8$.

Solve the simultaneous equations

$$\lfloor x \rfloor + \{y\} = 4.9$$

$$\{x\} + \lfloor y \rfloor = -1.4$$

for x and y.

Based on STEP 2021 Paper 2 Q3 (continued on next slide)

MEI

Continued…

Given that x, y and z satisfy the simultaneous equations

$$x + \lfloor y \rfloor + \{z\} = 3.9$$
$$\{x\} + y + \lfloor z \rfloor = 5.3$$
$$\lfloor x \rfloor + \{y\} + z = 5$$

show that $\{y\} + \lfloor z \rfloor = 3.2$ and solve the equations.

Solve the simultaneous equations

$$x + 2\lfloor y \rfloor + \{z\} = 3.9$$
$$\{x\} + 2y + \lfloor z \rfloor = 5.3$$
$$\lfloor x \rfloor + 2\{y\} + z = 5$$

Based on STEP 2021 Paper 2 Q3

Solutions

Because maths matters!

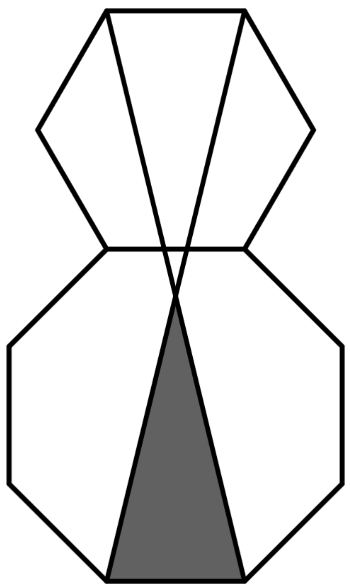
Octagon-hexagon problem

The problem

The regular hexagon and the regular octagon shown share a common edge.

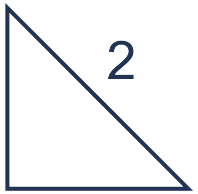
The length of that edge is 2 cm.

Find the shaded area.



Steps to the solution

There are two helpful right-angled triangles that can be used:



and

One is a $45°$- $45°$ triangle with height $\sqrt{2}$

The other is a $60°$- $30°$ triangle with height $\sqrt{3}$

The overall height is $2 + 2\sqrt{2} + 2\sqrt{3}$ units making the height of the shaded triangle $1 + \sqrt{2} + \sqrt{3}$. Since the base of the triangle is 2 units, the shaded area is $\frac{1}{2} \times 2 \times \left(1 + \sqrt{2} + \sqrt{3}\right)$.

This is also $1 + \sqrt{2} + \sqrt{3}$

MEI

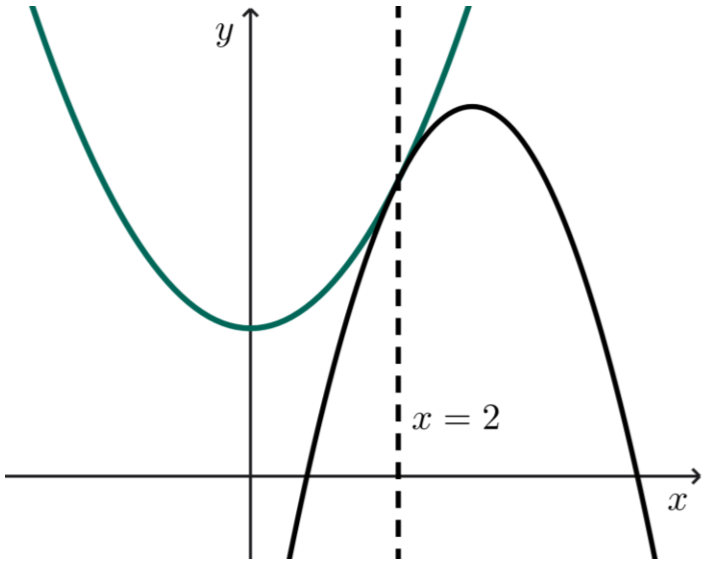
Common tangent problem

The problem

Two parabolas share a common tangent at $x = 2$.

The vertex of one of the parabolas is at (0,2) and the vertex of the other is at (3,5).

Find the equation of each parabola and the equation of the common tangent.



Steps to the solution

The equation of one parabola can be written as $y = ax^2 + 2$

The equation of the other parabola can be written as $y = -b(x-3)^2 + 5$

When $x = 2$, the equations should return the same y value so $4a + 2 = -b + 5$

The gradients are also the same when $x = 2$

$$y = ax^2 + 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2ax$$

$$y = -b(x-3)^2 + 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2b(x-3)$$

When $x = 2$, $4a = 2b$

Solving the equations simultaneously leads to the solution.

MEI

Incredible indices problem

The problem

Find the solutions for x to the equation below:

$$(x^2 + 2x - 14)^{(x^2 - 3x - 28)} = 1$$

Steps to the solution

There are different cases that could give rise to this equation

i) $x^2 - 3x - 28 = 0$

ii) $x^2 + 2x - 14 = 1$

iii) $x^2 + 2x - 14 = -1$

For i) the values of x need to be tested in $x^2 + 2x - 14$ to check that neither gives 0^0

For iii) the index would need to result in an even power of -1 for the equation to be true.

Annulus problem – Solution 1

Required area, A:

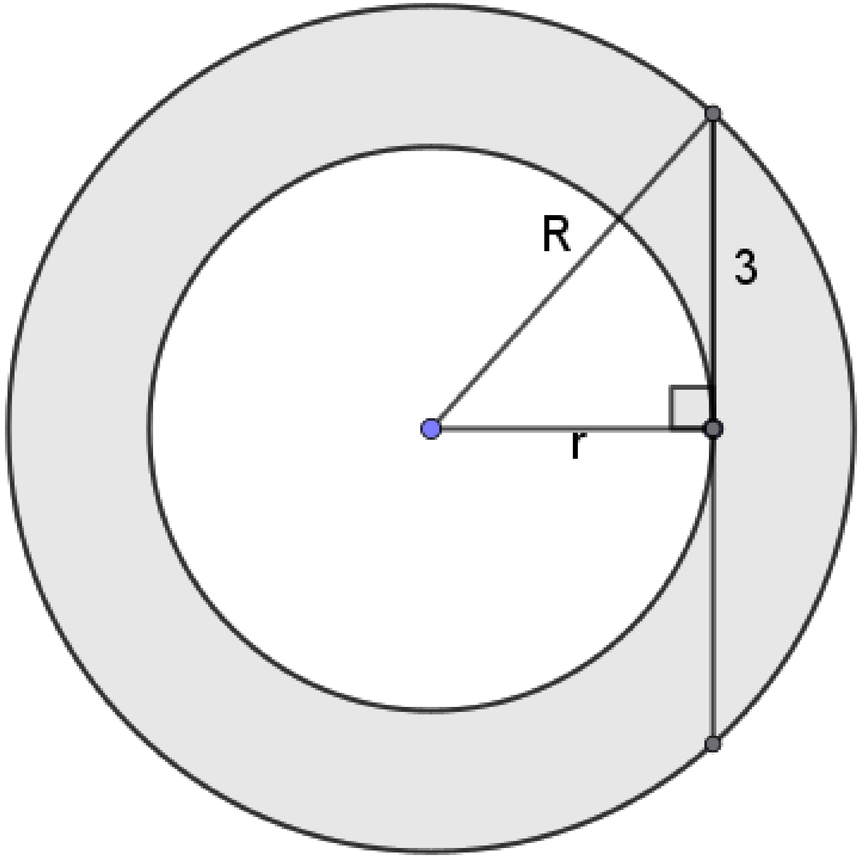
$$A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$$

Right angled triangle gives:

$$r^2 + 3^2 = R^2 \quad \Rightarrow \quad R^2 - r^2 = 9$$

Therefore required area is:

$$A = \pi(R^2 - r^2) = 9\pi \text{ cm}^2$$



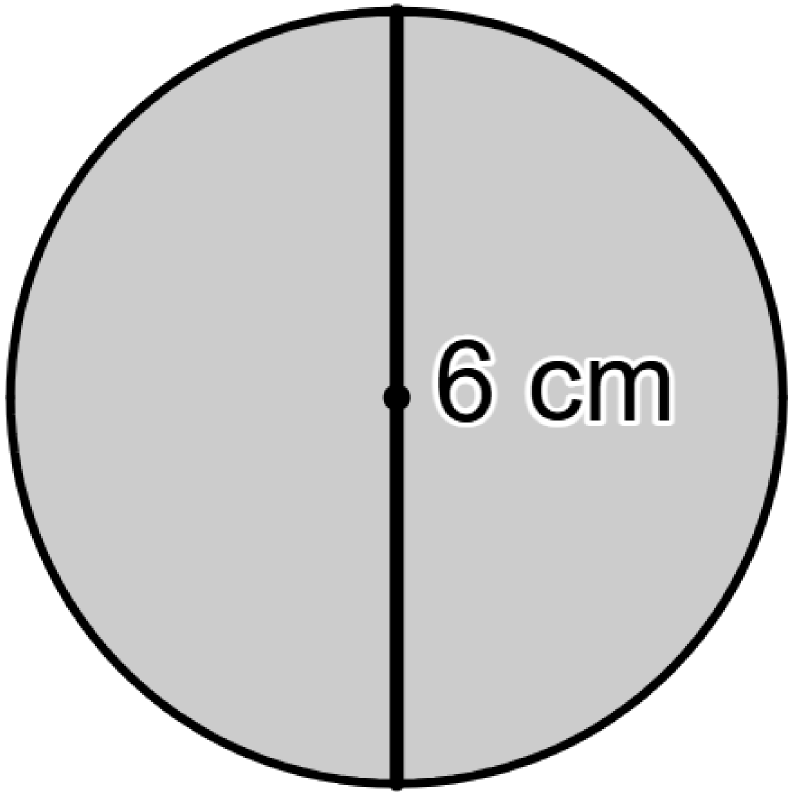
Annulus problem – Solution 2

Notice that the picture still holds when the centre circle has a radius of 0.

The circle then has a radius of 3 cm

Required area, A:

$$A = \pi(3)^2 = 9\ \mathrm{cm}^2$$



Tetromino problem solution

Imagine adding a chequerboard shading onto the grid.

Each tetromino placed on the grid would inherit a shading of some of its squares.

Six of the seven would have two out of four.

The ‘T’ brick would have either one or three.

Across the seven this would mean a total of either 13 or 15 shaded squares. Having 14 is impossible.

Therefore, there is no way they could be placed on the grid.

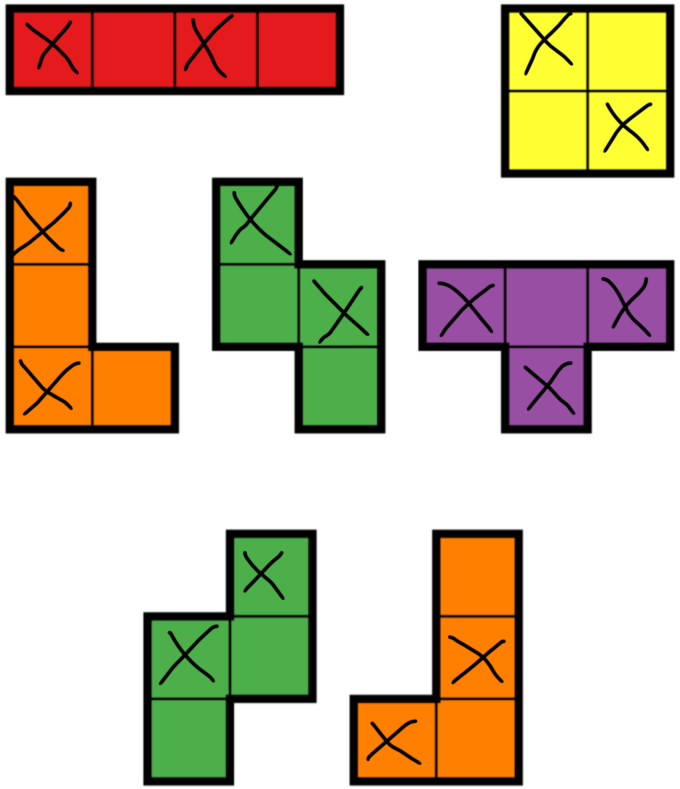
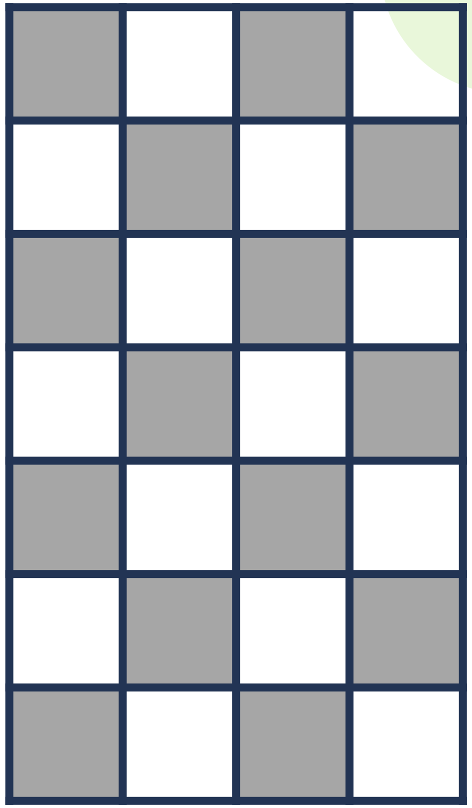


Image under creative commons

Tetromino Tiling 5x8.svg *by* R. A. Nonenmacher.



80-20-80 triangle solution

Many solutions to this problem exist. Also, many variants of 80-20-80 problems exist. I was shown this version of the problem by Colin Wright, a frequent MEI conference attendee.

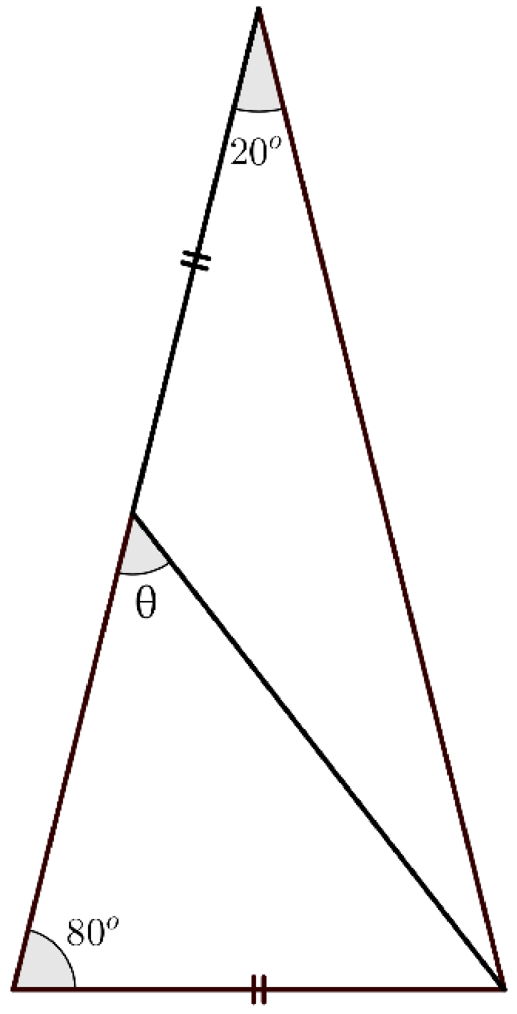
If you want a hint before the solution. Note that I believe all solutions involve that $80 - 20 = 60$ and then finding an equilateral triangle.

cut-the-knot.org maths has eleven (fairly distinct) solutions to this problem, including one I and, independently a student of mine I produced:

https://www.cut-the-knot.org/triangle/80-80-20/LongSol8.shtml

Other solutions (solution 2 is my favourite): https://www.cut-the-knot.org/triangle/80-80-20/IndexToLong.shtml

Other variants: https://www.cut-the-knot.org/triangle/80-80-20/index.shtml#Long



MEI

Not all nice problems are puzzles

Solve the simultaneous equations

$$\lfloor x \rfloor + \{y\} = 4.9$$
$$\{x\} + \lfloor y \rfloor = -1.4$$

for x and y.

From $\lfloor x \rfloor + \{y\} = 4.9$, $\lfloor x \rfloor$ must be 4 and $\{y\}$ must be 0.9

From $\{x\} + \lfloor y \rfloor = -1.4$, $\lfloor y \rfloor$ must be -2 and $\{x\}$ must be 0.6

Hence $x = 4 + 0.6 = 4.6$ and $y = -2 + 0.9 = -1.1$

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Continued…

$x + \lfloor y \rfloor + \{z\} = 3.9$ (1)
$\{x\} + y + \lfloor z \rfloor = 5.3$ (2)
$\lfloor x \rfloor + \{y\} + z = 5$ (3)

$(1) + (2) + (3)$ gives $2x + 2y + 2z = 14.2$
so $x + y + z = 7.1$

$(2) + (3)$ gives $x + y + z + \{y\} + \lfloor z \rfloor = 10.3$ so $\{y\} + \lfloor z \rfloor = 10.3 - 7.1 = 3.2$

$\lfloor z \rfloor = 3$ and $\{y\} = 0.2$

$(1) + (2)$ gives $x + y + z + \{x\} + \lfloor y \rfloor = 9.2$ so $\{x\} + \lfloor y \rfloor = 9.2 - 7.1 = 2.1$

$\lfloor y \rfloor = 2$ and $\{x\} = 0.1$

$(1) + (3)$ gives $x + y + z + \lfloor x \rfloor + \{z\} = 8.9$ so $\lfloor x \rfloor + \{z\} = 8.9 - 7.1 = 1.8$

$\lfloor x \rfloor = 1$ and $\{z\} = 0.8$

$x = 1 + 0.1 = 1.1$, $y = 2 + 0.2 = 2.2$, $z = 3 + 0.8 = 3.8$

Continued…

$x + 2\lfloor y \rfloor + \{z\} = 3.9$ (1)   (1) + (2) + (3) gives $2x + 4y + 2z = 14.2$

$\{x\} + 2y + \lfloor z \rfloor = 5.3$ (2)   so $x + 2y + z = 7.1$

$\lfloor x \rfloor + 2\{y\} + z = 5$ (3)

(1) + (2) $x + 2y + z + 2\lfloor y \rfloor + \{x\} = 9.2$ so $2\lfloor y \rfloor + \{x\} = 9.2 - 7.1 = 2.1$

$\{x\} = 0.1,\ 2\lfloor y \rfloor = 2 \Rightarrow \lfloor y \rfloor = 1$

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$\{z\} = 0.8,\ \lfloor x \rfloor = 1$

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$2\{y\} = 0.2$ so $\{y\} = 0.1$, or $\{y\} = 0.6$

For $\{y\} = 0.1$, $\lfloor z \rfloor = 3$, for $\{y\} = 0.6$, $\lfloor z \rfloor = 2$

Two possibilities: $x = 1.1, y = 1.1, z = 3.8$ or $x = 1.1, y = 1.6, z = 3.2$

Problem: 80-20-20

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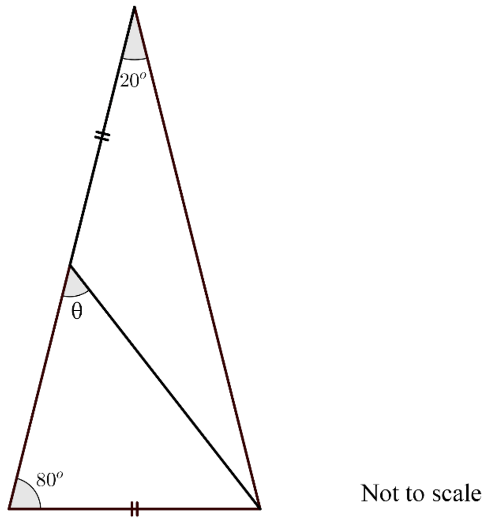

Ratings:

A picture helps	10	Try a simpler version	2
Hidden clue	8	Beautiful solution	4
Deeper meaning	3	Complicated maths	1

What have you learnt about problem solving?

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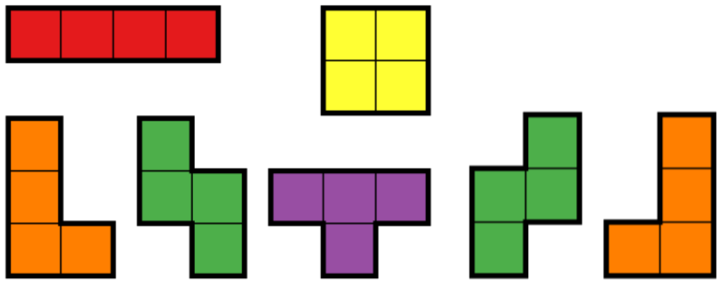

Image under creative commons Tetromino Tiling 5x8.svg *by* R. A. Nonenmacher

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Problem: Annulus

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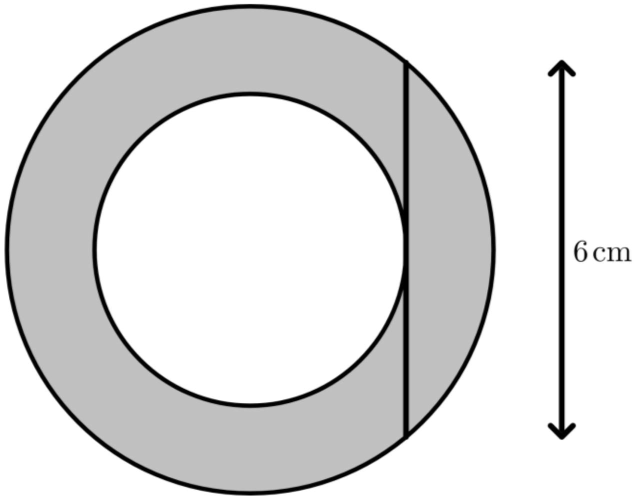

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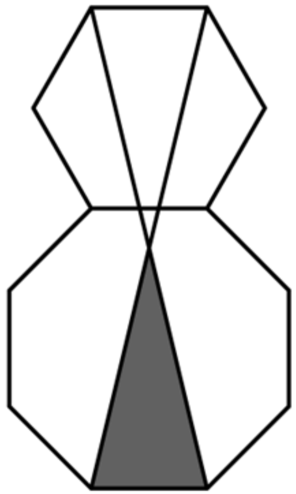
Problem: Octagon-hexagon



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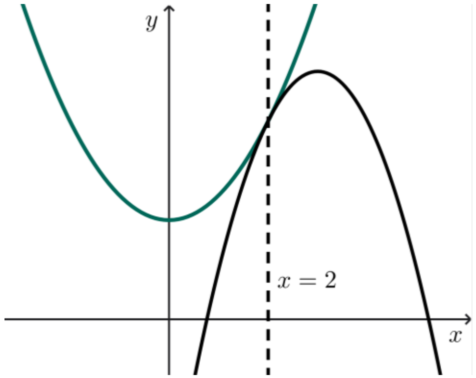
Problem: Common tangent



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