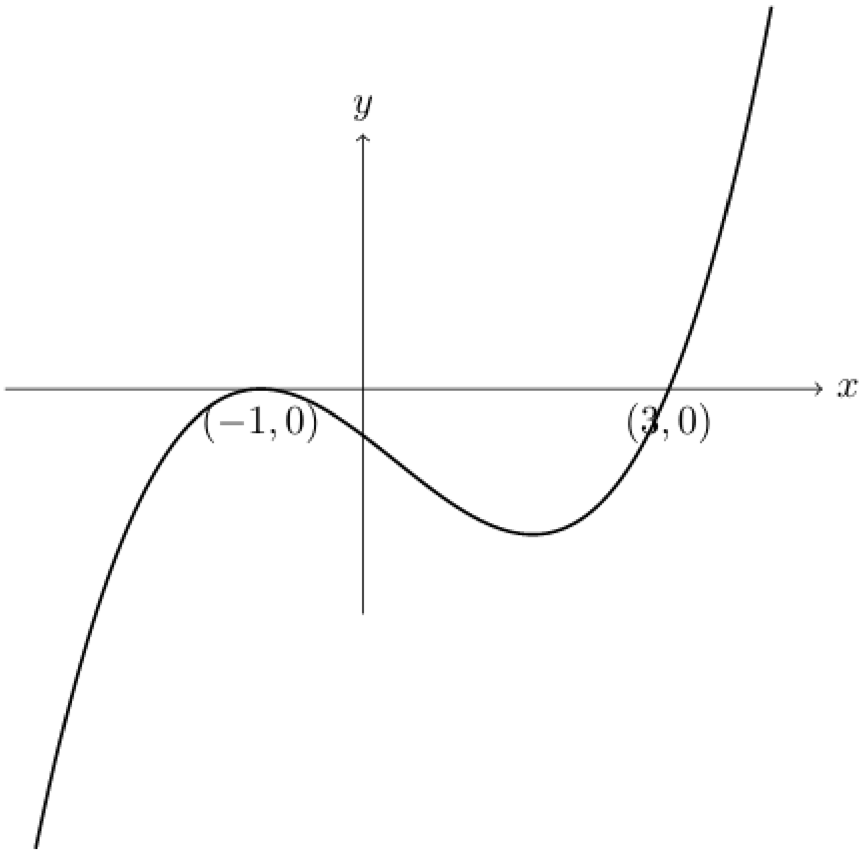
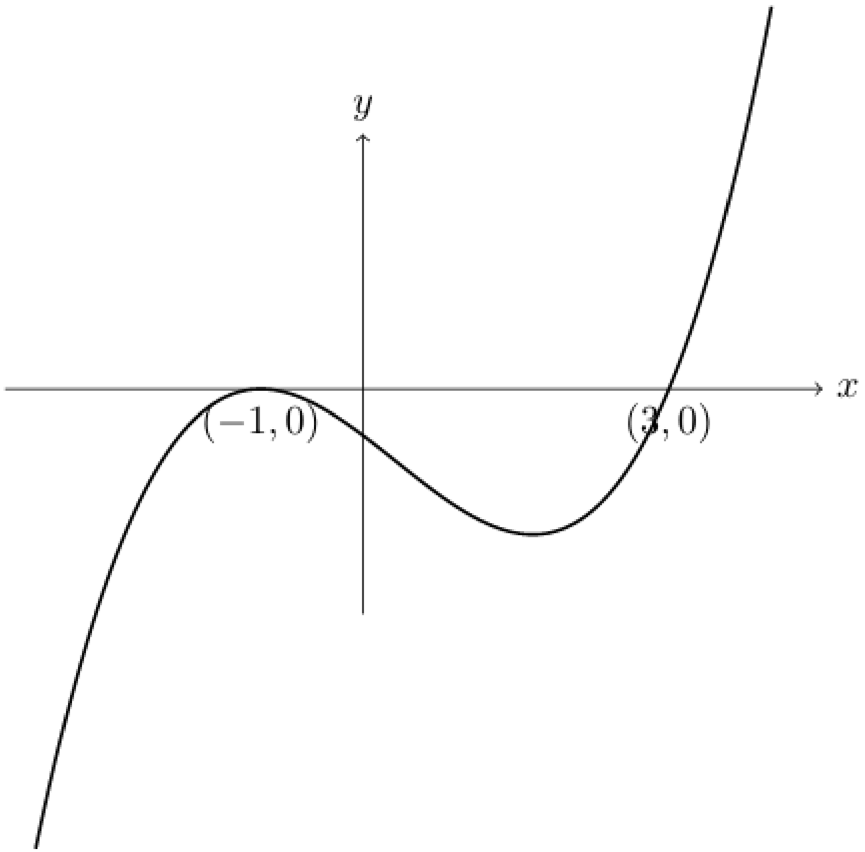
Embedding Self-Explanation training into the curriculum

EXETER MATHS SCHOOL

MEI
Because maths matters!



y
x
$(-1, 0)$
$(3, 0)$



The diagram shows the curve $y = x^3 + ax^2 + bx + c$.

The curve passes through the point $(3, 0)$ and has a local maximum at $(-1, 0)$.

Determine the values of a, b and c.

Student approach

Looks for a method

Starts calculating

Uses procedures

Waits for the next step

Expert approach

Looks for mathematical structure

Interprets information first

Connects ideas

Predicts what comes next

Self-Explanation Training

By training students to read and think like experienced mathematicians, we can improve their understanding of abstract concepts and create the logical connections needed to solve complex problems.

Shown to be effective in many other subjects outside of maths (e.g., Chi et al., 1989, Ainsworth and Burcham, 2007)

What it is

After reading each question or line of mathematics:

- Try to identify and elaborate upon the main ideas used.

- After each line of a question, try and do some mathematics before reading on. You can usually turn the words into algebra or use formulas, equations, or other ideas to make the question much easier.

- Attempt to explain the mathematics in terms of previous ideas. These may be ideas from the information in the question or text, examples from previous questions or mathematics you have covered, or ideas from your own prior knowledge of the topic area.

Teacher vs Student

Definition. An abundant number is a positive integer n whose divisors add up to more than $2n$. For example, 12 is abundant because $1 + 2 + 3 + 4 + 6 + 12 > 24$.

Theorem. The product of two distinct primes is not abundant.

(L1) **Proof.** Let $n = p_1 p_2$ where p_1 and p_2 are distinct primes.

(L2) Assume that $2 \leq p_1$ and $3 \leq p_2$.

(L3) The divisors of n are 1, p_1, p_2 and $p_1 p_2$.

(L4) Note that $\dfrac{p_1 + 1}{p_1 - 1}$ is a decreasing function of p_1.

(L5) So $\max \left\{ \dfrac{p_1 + 1}{p_1 - 1} \right\} = \dfrac{2 + 1}{2 - 1} = 3$.

(L6) Hence $\dfrac{p_1 + 1}{p_1 - 1} \leq p_2$.

(L7) So $p_1 + 1 \leq p_1 p_2 - p_2$.

(L8) So $p_1 + 1 + p_2 \leq p_1 p_2$.

(L9) So $1 + p_1 + p_2 + p_1 p_2 \leq 2 p_1 p_2$.

1. Restating our conditions and setting out the variable n

Setting up a general case of 2 distinct primes Calling it n. Notice all primes are positive so n is a positive integer.

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2. This statement can be made WLOG as all sets of two primes can be ordered to satisfy these conditions

Since distinct primes, one must be smaller than the other. The two smallest primes are 2 and 3 so one must be bigger or equal to 3 & one bigger or equal to 2

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3. Primes have no factors so we can state the factors of n

p_1 + p_2 prime so only divisors are 1 + themselves, so product $n=p_1p_2$ can only have divisors $1, p_1, p_2$ + p_1p_2 (n itself)

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4. If $\frac{p_1+1}{p_1-1}$ is a decreasing function we can set an upper bound on it

If $y = \dfrac{p_1+1}{p_1-1} = \dfrac{p_1-1+2}{p_1-1} = 1 + 2(p_1-1)^{-1}$

then $\dfrac{dy}{dp_1} = -2(p_1-1)^{-2} = \dfrac{-2}{(p_1-1)^2} < 0$ since $-2<0$, $(p_1-1)^2>0$

so y decreasing function of p_1

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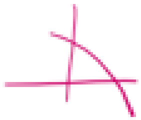
5. Here we find this upper bound, given our assumption about p_1 from line 2

A decreasing function can only go down so smallest possible value of p_1 will give largest possible value of $y = \frac{p_1+1}{p_1-1}$. We know $p_1 \geq 2$ so smallest possible value of p_1 is 2, so biggest possible value of y is $\frac{2+1}{2-1} = \frac{3}{1} = 3$



The mathematics changes…

- Proof
- Geometry
- Mechanics
- Statistics

…but the thinking doesn't.

Experts consistently ask:

- What mathematical ideas are here?
- What do I already know?
- Why has this information been given?
- What do I predict comes next?
- Self-Explanation Training helps students internalise these questions until they ask them automatically.

14 Fig. 14 shows a circle with centre O and radius r cm. The chord AB is such that angle AOB $= x$ radians. The area of the shaded segment formed by AB is 5% of the area of the circle.

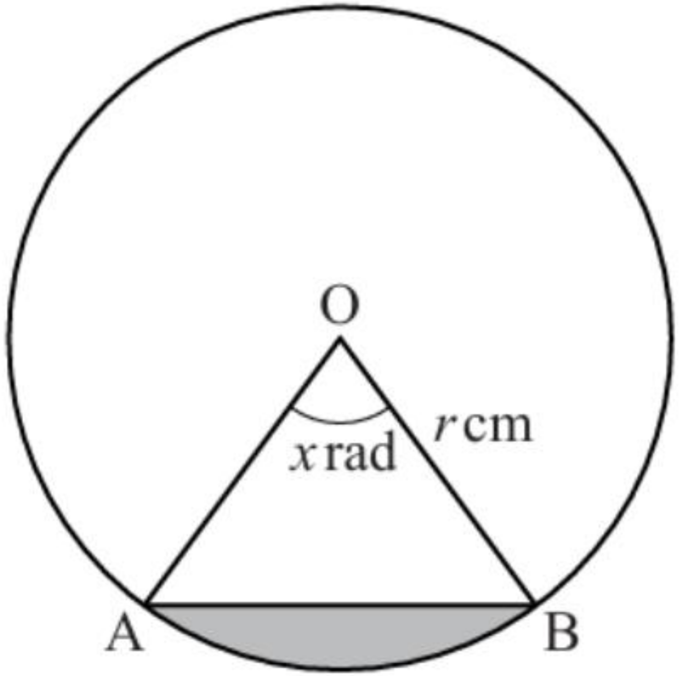


Fig. 14

OCR B - H640/01-2019

(a) Show that $x - \sin x - \frac{1}{10}\pi = 0$. **[4]**

Line of question	Student thinking	Expert self-explanation	Teacher prompt
Circle with centre O and radius r.	Try to assign a value to r or think there isn't enough information.	The radius is unknown, so I should expect it to cancel out. This is probably about expressing areas algebraically rather than finding numerical values.	What quantities are unknown? Which of them do you think will eventually disappear?
Angle $AOB = x$ radians.	Use the degree formula or forget the formula for sector area in radians.	The angle is given in radians, so I should be thinking of the sector area formula $\frac{1}{2}r^2x$, not degrees.	Why has the angle been given in radians rather than degrees?
The shaded region is a segment.	Confuse a segment with a sector or use only one area formula.	A segment is the area of the sector minus the triangle. I need to identify both parts before calculating anything.	How is a segment formed? Which two shapes make it up?
Area of the shaded segment is 5% of the circle.	Calculate 5% incorrectly or forget to relate the two expressions.	I can write the segment area in two different ways: $\frac{1}{2}r^2(x - \sin x)$ and $0.05\pi r^2$.	Can you write two expressions for the same area?
Show that $x - \sin\ x - \frac{\pi}{10} = 0$.	Continue manipulating with r or think another formula is needed.	The required result contains only x, so the r^2 terms must cancel when I equate the two area expressions	The answer contains no r. What does that tell you about your algebra?
Form the equation.	Rearrange too early or make algebraic errors before cancelling common factors.	Equate the two expressions for the segment area, simplify carefully, and rearrange into the required form	Before simplifying, have you written two equivalent expressions for the same area?

Your role has changed

As learners you generated self-explanations.

As teachers your role is to design opportunities for students to generate them.

Identify:

- where students stop thinking
- what an expert notices
- the question that will help students make that connection

15 A model for the motion of a small object falling through a thick fluid can be expressed using the differential equation

$$\frac{dv}{dt} = 9.8 - kv,$$

where $v\,\mathrm{m\,s^{-1}}$ is the velocity after t s and k is a positive constant.

(a) Given that $v = 0$ when $t = 0$, solve the differential equation to find v in terms of t and k. **[7]**

(b) Sketch the graph of v against t. **[2]**

Experiments show that for large values of t, the velocity tends to $7\,\mathrm{m\,s^{-1}}$.

(c) Find the value of k. **[2]**

(d) Find the value of t for which $v = 3.5$. **[1]**

Question information	Expert self-explanation	Likely misconception	Teacher prompt
A particle falls through a fluid.	I should think about the physical meaning as well as the mathematics.	Ignore the context.	What is changing? What do the variables represent?
$\frac{dv}{dt} = 9.8 - kv$	This is a first-order separable differential equation. The resisting force increases with velocity.	Try to integrate immediately.	How do you recognise the equation type?
(v=0) when (t=0)	This initial condition determines the integration constant.	"I'll use it later" without understanding why.	Why has the examiner given this information now?
Sketch the graph.	Velocity increases but the rate of increase decreases because resistance grows.	Need the exact solution first.	What qualitative features can you predict?
Velocity tends to 7.	Terminal velocity means acceleration becomes zero.	Substitute into the solution.	What does terminal velocity mean mathematically?
Find (k).	Use (dv/dt=0) when (v=7). No further integration is needed.	Repeat all previous calculations.	Can you answer this using the differential equation directly?

12 The jaguar is a species of big cat native to South America. Records show that 6% of jaguars are born with black coats. Jaguars with black coats are known as black panthers. Due to deforestation a population of jaguars has become isolated in part of the Amazon basin. Researchers believe that the percentage of black panthers may not be 6% in this population.

(a) Find the minimum sample size needed to conduct a two-tailed test to determine whether there is any evidence at the 5% level to suggest that the percentage of black panthers is not 6%. **[3]**

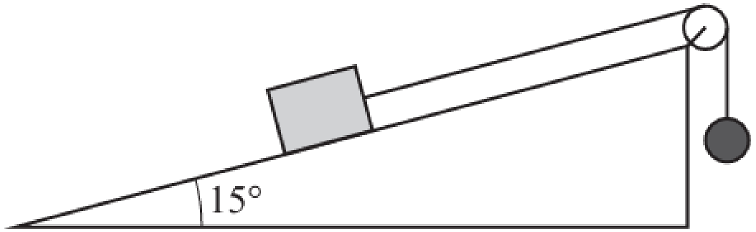
A research team identifies 70 possible sites for monitoring the jaguars remotely. 30 of these sites are randomly selected and cameras are installed. 83 different jaguars are filmed during the evidence gathering period. The team finds that 10 of the jaguars are black panthers.

(b) Conduct a hypothesis test to determine whether the information gathered by the research team provides any evidence at the 5% level to suggest that the percentage of black panthers in this population is not 6%. **[7]**

Question information	Expert self-explanation	Likely misconception	Teacher prompt
6% are black.	This is the population proportion, so it becomes (H_0).	Ignore it until later.	What does this tell us about the null hypothesis?
Researchers believe the percentage may not be 6%.	"May not be" means a two-tailed test.	Write a one-tailed alternative.	Which word determines the alternative hypothesis?
Random sample.	Randomness supports the assumptions of the model.	Ignore the statement.	Why is random sampling important?
83 jaguars observed.	(n=83).	Just another number.	Which symbol represents this value?
10 are black.	(x=10).	Confuse with the proportion.	What does this number represent?
Test at the 5% significance level.	Compare the p-value (or critical region) with 0.05.	Treat 5% as the probability the null is true.	What does the significance level actually represent?

13 A block of mass 8 kg is placed on a rough plane inclined at 15° to the horizontal. The coefficient of friction between the block and the plane is 0.3.

One end of a light rope is attached to the block. The rope passes over a smooth pulley fixed at the top of the plane, and a sphere of mass 5 kg, attached to the other end of the rope, hangs vertically below the pulley. The part of the rope between the block and the pulley is parallel to the plane. The system is released from rest, and as the sphere falls the block moves directly up the plane with acceleration $a\,\text{m s}^{-2}$.



(a) On the diagram in the Printed Answer Booklet, show all the forces acting on the block and on the sphere. **[4]**

(b) Write down the equation of motion for the sphere. **[2]**

(c) Determine the value of a. **[6]**

Question information	Expert self-explanation	Likely misconception	Teacher prompt
Rough plane	Friction must act.	Forget friction.	What forces exist before reading further?
Inclined at angle (\theta)	I'll need to resolve weight into components.	Resolve incorrectly.	Why do we resolve forces?
Projected up the plane	Friction acts down the plane because it opposes motion.	Friction points up the plane.	What determines the direction of friction?
Coefficient of friction (\mu)	Friction equals (\mu R), so I must first find the reaction force.	Use (\mu mg).	What does friction depend on?
Constant deceleration	Acceleration is opposite the direction of motion.	Ignore the sign convention.	How will you define your positive direction?
Find (\mu).	I'll probably resolve perpendicular first to find (R).	Start substituting numbers.	Which equation is easiest to form first? Why?

10 The screenshot in Fig. 10 shows the probability distribution for the continuous random variable X, where $X \sim \mathrm{N}(\mu, \sigma^2)$.

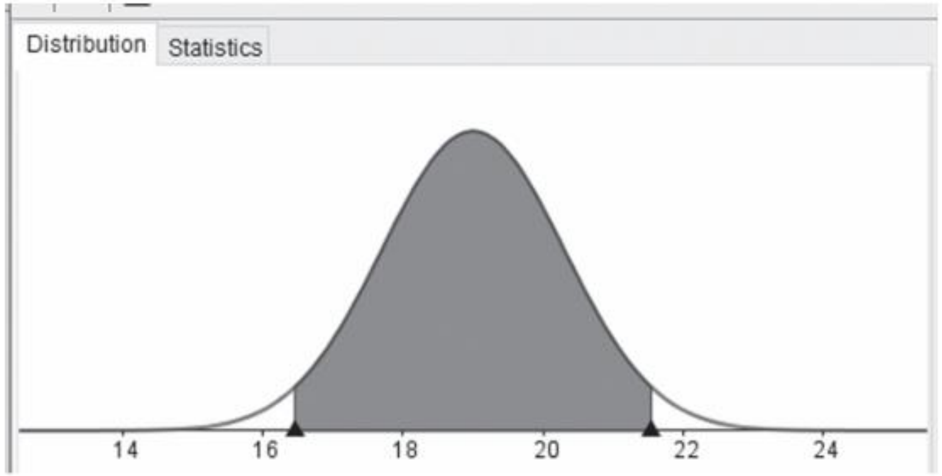


Fig. 10

The area of each of the unshaded regions under the curve is 0.025. The lower boundary of the shaded region is at 16.452 and the upper boundary of the shaded region is at 21.548.

(i) Calculate the value of μ. **[1]**

(ii) Calculate the value of σ^2. **[3]**

Line of question	Student thinking	Expert self-explanation	Teacher prompt
$X \sim N(\mu, \sigma^2)$	Ignore the notation or think it only tells me which formula to use.	This tells me the distribution is normal, so it is symmetric about the mean. The centre of the graph is μ.	What information does the notation tell us before we calculate anything?
Each unshaded region has area 0.025.	Add the areas incorrectly or fail to recognise the 95% interval.	Each tail contains 2.5%, so the shaded region is the middle 95% of the distribution.	What proportion of the distribution is shaded? Why?
Lower boundary is 16.452 and upper boundary is 21.548.	Treat the two values independently without recognising the symmetry.	These are the boundaries of the central 95%, so they should be equally spaced about the mean because the normal distribution is symmetric.	What relationship should these two numbers have in a normal distribution?
Calculate μ.	Reach immediately for the calculator.	The mean must lie exactly halfway between the two boundaries. I don't need any probability calculations yet.	Can you answer part (i) without using a calculator?
Calculate σ^2.	Forget the critical value of 1.96 or use the full width instead of half the interval.	The boundaries correspond to $z = \pm 1.96$. The distance from the mean to either boundary equals 1.96σ.	Which standardised value corresponds to the central 95%?
Using the boundaries	Square too early, forget to square at the end, or substitute both boundaries unnecessarily.	First calculate the distance from the mean to one boundary, then divide by 1.96 to find σ, and finally square it to obtain σ^2.	Which quantity are you finding first—σ or σ^2? Why?

What did you notice?

Planning your next lesson

Key point – “START SMALL” **Don't redesign a scheme of work. Modify one worked example next week.**

Your challenge

- Choose one worked example from next week's lessons.
- Find three places where students typically switch into **procedure mode**.
- Write one self-explanation prompt for each.
- Try it next week.

Looks for a method

Starts calculating

Uses procedures

Waits for the next step

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